1. Discrete-Time Signals and Systems. Summary

1.1. Discrete-Time Signals and Systems. Basic Definitions

1.1.1. Discrete and Digital Signals

1.1.1.1. Basic Definitions

Signals may be classified into four categories depending on the characteristics of **the time-variable** and **values** they take:

Signals	***Time***	***Descriptions***	***Notes***
Continuous-time (analogue)	Defined for every value of time	Functions of a continuous variable $f(t)$	They take on values in the continuous interval (a,b), $a,b \to \infty$
Discrete-time	Defined only at discrete values of time	Sequences of real or complex numbers, $f(nT) = f(n)$	They take on values in the continuous interval (a,b), $a,b \to \infty$ Sampling process Sampling interval, period: T Sampling rate: samples per second Sampling frequency (Hz): $f_S = 1/T$

Signals	***Value***	***Descriptions***	***Notes***
Continuous-valued	They can take all possible values on finite or infinite range	Functions of a continuous variable or sequences of numbers	Defined for every value of time or Only at discrete values of time
Discrete-valued	They can take on values from a finite set of possible values	Functions of a continuous variable or sequences of numbers	Defined for every value of time or only at discrete values of time

Digital filter theory:

Signals	***Definition and description***	***Notes***
Discrete-time	Defined only at discrete values of time and they can take all possible values on finite or infinite range. Sequences of real or complex numbers.	Sampling process
Digital	Discrete-time and discrete-valued signals (i.e. discrete - time signals taking on values from a finite set of possible values)	Sampling, quantizing and coding process i.e. analogue-to-digital conversion

1.1.1.2. Discrete-Time Signal Representations

A. Functional representations:

$$x(n) = \begin{cases} 1 & for \quad n = 1,3 \\ 6 & for \quad n = 2 \\ 0 & elsewhere \end{cases}$$

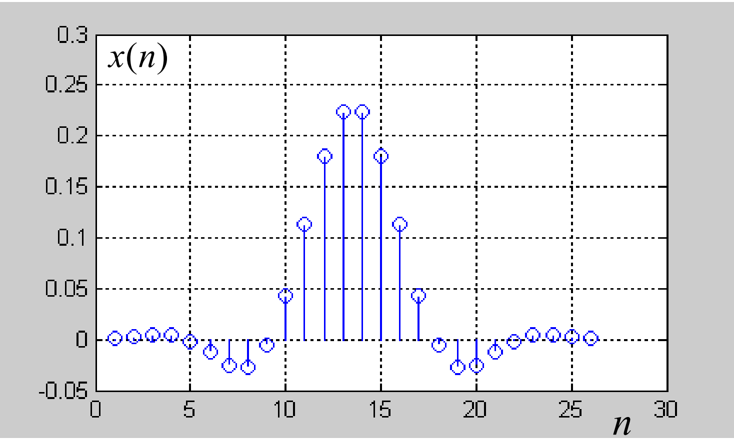
B. Tabular representation:

n	...	-2	-1	0	1	2	...
$x(n)$	...	0	1.3	2.8	-1.0	-0.4	...

C. Sequence representation:

$$x(n) = \{\dots\ 0\ \ 1.3\ \ 2.8\ \ -1.0\ \ -0.4\ \ \dots\}$$

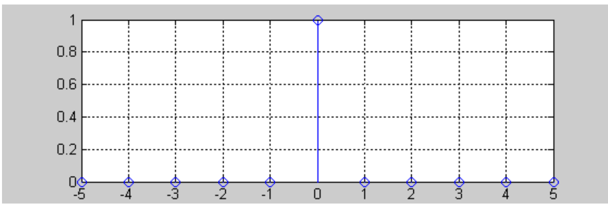
D. Graphical representation:



1.1.1.3. Some Elementary Discrete-Time Signals

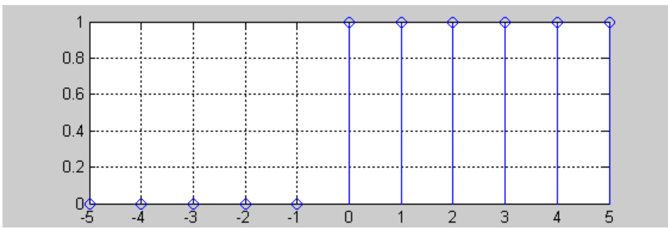
A. Unit sample sequence (unit sample, unit impulse, unit impulse signal)

$$\delta(n) = \begin{cases} 1 & for \quad n = 0 \\ 0 & for \quad n \neq 0 \end{cases}$$



B. Unit step signal (unit step, Heavisede step sequence)

$$u(n) = \begin{cases} 1 & for \quad n \geq 0 \\ 0 & for \quad n < 0 \end{cases}$$



C. Complex-valued exponential signal (complex sinusoidal sequence, complex phasor, complex-valued function)

$x(n) = e^{j\omega t}$ where $\omega, t \notin R$ and $j = \sqrt{-1}$ (imaginary unit)

$|x(n)| = 1$ and $\arg[x(n)] = \omega t$

1.1.2. Discrete-Time System. Definition

A discrete-time system is a device or algorithm that operates on a discrete signal called *the input* or *excitation*, according to some rule to produce another discrete-time signal called *the output* or *response*.

We say that the input signal $x(t)$ is transformed by the system into a signal $y(t)$ and express the general relationship between $x(t)$ and $y(t)$ as

$$y(n) \equiv H[x(n)]$$

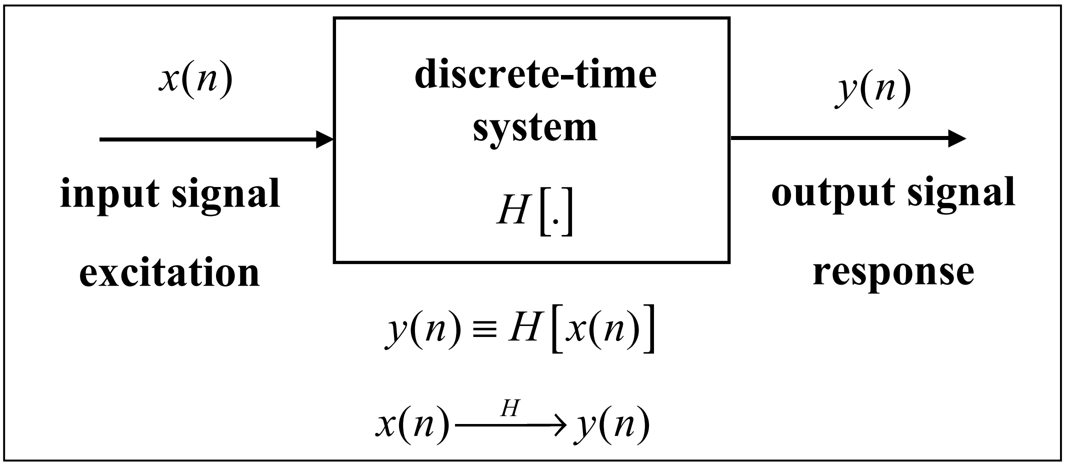
where the symbol denotes the transformation $H[.]$ (also called operator or mapping) or processing performed by the system on $x(n)$ to produce $y(n)$.

The input-output description of a discrete-time system consists of a mathematical expressions or rules, which explicitly done the relations between the input and output signals (so-called *input-output relationships*). The system can be assumed to be a "black box" to the user.

Input-output relationship description:

$$y(n) \equiv H[x(n)]$$

$$x(n) \xrightarrow{H} y(n)$$



1.1.3. Classification of Discrete-Time Systems

1.1.3.1. Static vs. Dynamic Systems. Definition

A discrete-time system is called *static* or *memoryless* if its output at any instant n depends at most on the input sample at the same time, but not past or future samples of the input. In the other case, the system is said to be *dynamic* or to have *memory*.

If the output of a system at time n is completely determined by the input samples in the interval from $n-N$ to n ($N \geq 0$), the system is said to have memory of *duration* N .

If $N = 0$, the system is *static* or *memoryless.*
If $0 < N < \infty$, the system is said to have *finite memory*.
If $N \to \infty$, the system is said to have in*finite memory*.

Examples:

The static (memoryless) system: $y(n) = nx(n) + bx^3(n)$

The dynamic system with finite memory:

$$y(n) = nx(n) + bx^3(n-1) \qquad y(n) = \sum_{k=0}^{N} h(k)x(n-k)$$

The dynamic system with infinite memory:

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$$

1.1.3.2. Time-Invariant vs. Time-Variable Systems. Definition

A discrete-time system is called *time-invariant* if its input-output characteristics do not change with time. In the other case, the system is called *time-variant*.

Definition. A relaxed system $H[.]$ is *time-invariant* or *shift-invariant* if only if $x(n) \xrightarrow{H} y(n)$ implies that $x(n-k) \xrightarrow{H} y(n-k)$ for every input signal $x(n)$ and every time shift k.

Examples:

The time-invariant system:

$$y(n) = x(n) + bx^3(n) \qquad y(n) = \sum_{k=0}^{N} h(k)x(n-k)$$

The time-variant system:

$$y(n) = nx(n) + bx^3(n-1) \qquad y(n) = \sum_{k=0}^{N} h^{N-n}(k)x(n-k)$$

1.1.3.3. Linear vs. Non-linear Systems. Definition

A discrete-time system is called *linear* if it satisfies the *linear superposition principle*. In the other case, the system is called *non-linear*.

Definition. A relaxed system $H[.]$ is *linear* if only if

$$H\left[a_1x_1(n) + a_2x_2(n)\right] = a_1H\left[x_1(n)\right] + a_2H\left[x_2(n)\right]$$

for any arbitrary input sequences $x_1(n)$ and $x_2(n)$, and any arbitrary constants a_1 and a_2.

The multiplicative (scaling) property of a linear system:

$$H\left[a_1x_1(n)\right] = a_1H\left[x_1(n)\right]$$

The additivity property of a linear system:

$$H[x_1(n)+x_2(n)] = H[x_1(n)] + H[x_2(n)]$$

Examples:

The linear system:

$$y(n) = \sum_{k=0}^{N} h(k)x(n-k) \qquad y(n) = x(n^2) + bx(n-k)$$

The non-linear system:

$$y(n) = nx(n) + bx^3(n-1) \qquad y(n) = \sum_{k=0}^{N} h(k)x(n-k)x(n-k+1)$$

1.1.3.4. Causal vs. Noncausal Systems. Definition

Definition. A system is said to be *causal* if the output of the system at any time n (i.e., $y(n)$) depends only on present and past inputs (i.e., $x(n), x(n-1), x(n-2), \ldots$). In mathematical terms, the output of a causal system satisfies an equation of the form

$$y(n) = F[x(n), x(n-1), x(n-2), \cdots]$$

where is $F[.]$ some arbitrary function.

If a system does not satisfy this definition, it is called *noncausal*.

Examples:

The causal system:

$$y(n) = \sum_{k=0}^{N} h(k)x(n-k) \qquad y(n) = x(n^2) + bx(n-k)$$

The noncausal system:

$$y(n) = nx(n+1) + bx^3(n-1) \qquad y(n) = \sum_{k=-10}^{10} h(k)x(n-k)$$

1.1.3.5. Stable vs. Unstable of Systems. Definition

Definition. An arbitrary relaxed system is said to be bounded input - bounded output (BIBO) stable if and only if every bounded input produces the bounded output. It means, that there exist some finite numbers say M_x and M_y, such that

$$|x(n)| \le M_x \le \infty, \Rightarrow |y(n)| \le M_y \le \infty$$

for all n. If some bounded input sequence $x(n)$, the output is unbounded (infinite); the system is classified as unstable.

Examples:

The stable system:

$$y(n) = \sum_{k=0}^{N} h(k)x(n-k) \quad y(n) = x(n^2) + 3x(n-k)$$

The noncausal system: $y(n) = 3^n x^3(n-1)$

1.1.3.6. Recursive vs. Nonrecursive Systems. Definitions

A system whose output $y(n)$ at time n depends on any number of the past outputs values $y(n-1)$, $y(n-2)\ldots$ is called a recursive system. Then, the output of a causal recursive system can be expressed in general as

$$y(n) = F\left[y(n-1), y(n-2), \ldots, y(n-N), x(n), x(n-1), \ldots, x(n-M)\right]$$

In contrast, if $y(n)$ at time n depends only on the present and past inputs, then

$$y(n) = F\left[x(n), x(n-1), \ldots, x(n-M)\right]$$

Such a system is called nonrecursive.

1.2. Linear-Discrete Time-Invariant System (LTI)

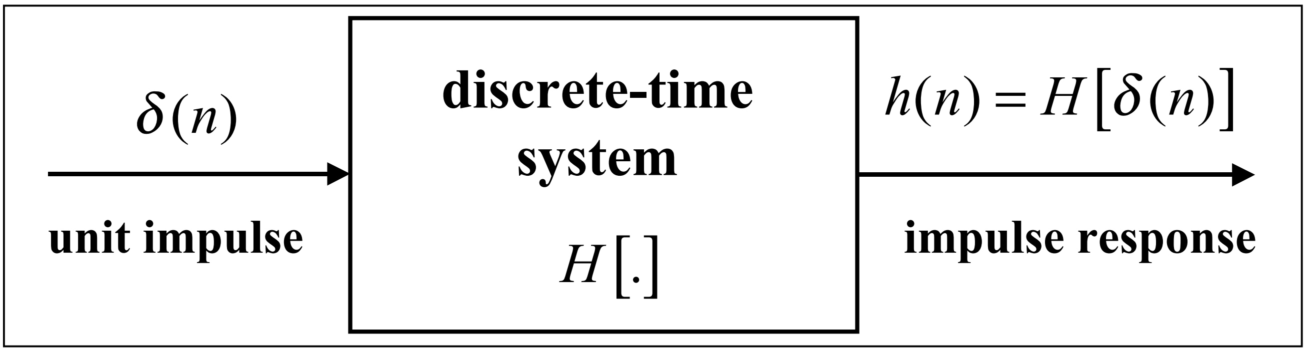
1.2.1. Time-Domain Representation

1.2.1.1 Impulse Response and Convolution, Convolution Sum

Unit impulse: $\delta(n)$

LTI: $H[.]$

(Unit) impulse response: $h(n) = H\left[\delta(n)\right]$



LTI description by convolution (convolution sum):

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = h(n) * x(n) = x(n) * h(n)$$

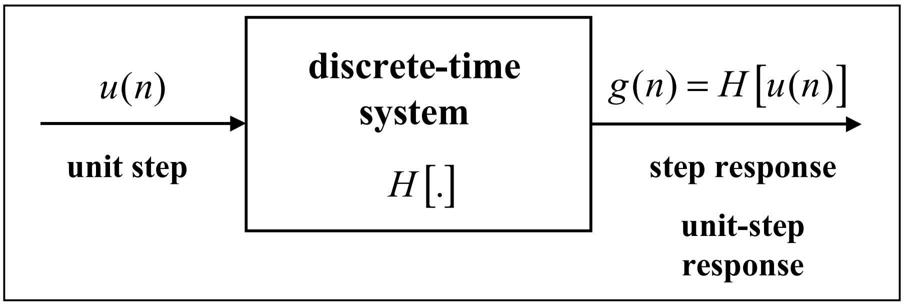
Viewed mathematically, the convolution operation satisfies the commutative law.

1.2.1.2. Step Response

Unit step: $u(n)$

LTI: $H[.]$

Step response (unit-step response): $g(n) = H[u(n)]$



$$s(n) = \sum_{k=-\infty}^{\infty} h(k)u(n-k) = \sum_{k=-\infty}^{n} h(k)$$

This expression relates the impulse response to the step response of the system.

Note:

$$s(n) = \sum_{k=-\infty}^{n} h(k) = h(n) + \sum_{k=-\infty}^{n-1} h(k) = h(n) + s(n-1)$$

$$h(n) = s(n) - s(n-1)$$

$$y(n) = \sum_{k=-\infty}^{n-1} x(k)\left[s(n-k) - s(n-k-1)\right]$$

1.2.2. Classification of LTI System

1.2.2.1. Causal LTI Systems

A relaxed LTI system is causal if and only if its impulse response is zero for negative values of n, i.e.

$$h(n) = 0 \quad for \quad n < 0$$

Then for the causal LTI systems is valid:

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k) = \sum_{k=-\infty}^{n} x(k)h(n-k)$$

1.2.2.2. Stable LTI Systems

A LTI is stable if its impulse response is absolutely summable, i.e.

$$\sum_{k=-\infty}^{\infty} |h(k)|^2 < \infty$$

1.2.2.3. Finite Impulse Response (FIR) LDTS and Infinite Impulse Response (IIR) LDTS

(Causal) FIR LTI systems: $y(n) = \sum_{k=0}^{N} h(k)x(n-k)$

(IIR) LTI systems: $y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$

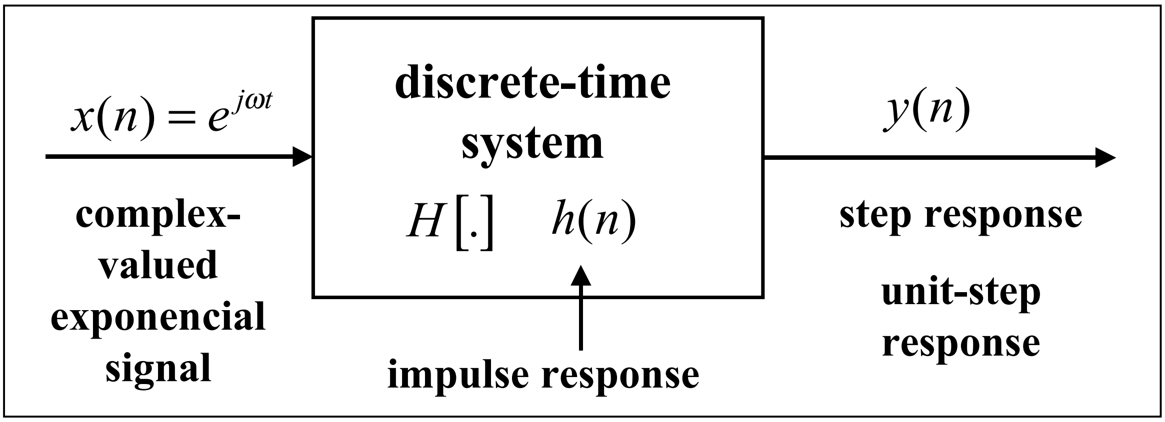
1.2.2.4. Recursive and Nonrecursive LTI Systems

Causal nonrecursive LTI: $y(n) = \sum_{k=0}^{N} h(k)x(n-k)$

Causal recursive LTI: $y(n) = \sum_{k=0}^{N} b(k)x(n-k) - \sum_{k=1}^{M} a(k)x(n-k)$

LTI systems characterized by Constant-Coefficient Difference Equations

1.3. Frequency-Domain Representation of Discrete Signals and LDTS



LTI system: $y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$

The impulse response: $h(n)$

Complex-valued exponential signal: $x(n) = e^{j\omega t}$ where $\omega, t \notin R$ and $j = \sqrt{-1}$ (imaginary unit)

LTI system output:

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) = \sum_{k=-\infty}^{\infty} h(k)e^{j\omega(n-k)} = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}e^{j\omega n} = e^{j\omega n}\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}$$

$$y(n) = e^{j\omega n}H(e^{j\omega})$$

Frequency response: $H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}$

$$H(e^{j\omega}) = \left|H(e^{j\omega})\right| e^{j\phi(\omega)}$$

$$H(e^{j\omega}) = \mathrm{Re}\left[H(e^{j\omega})\right] + j\,\mathrm{Im}\left[H(e^{j\omega})\right]$$

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)\cos\omega k - j\sum_{k=-\infty}^{\infty} h(k)\sin\omega k$$

The real component of $H(e^{j\omega})$: $\mathrm{Re}\left[H(e^{j\omega})\right] = \sum_{k=-\infty}^{\infty} h(k)\cos\omega k$

The imaginary component of $H(e^{j\omega})$: $\mathrm{Im}\left[H(e^{j\omega})\right] = -j\sum_{k=-\infty}^{\infty} h(k)\sin\omega k$

Magnitude response: $\left|H(e^{j\omega})\right| = \sqrt{\mathrm{Re}\left[H(e^{j\omega})\right]^2 + \mathrm{Im}\left[H(e^{j\omega})\right]^2}$

Phase response: $\phi(\omega) = \arg\left[H(e^{j\omega})\right] = arctg\dfrac{\mathrm{Im}\left[H(e^{j\omega})\right]}{\mathrm{Re}\left[H(e^{j\omega})\right]}$

Group delay function: $\tau(\omega) = -\dfrac{d\phi(\omega)}{d\omega}$

1.3.1. Comments on Relationship Between the Impulse Response and Frequency Response

An important property of

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}$$

is that this function is periodic with period 2π ($H(e^{j\omega}) = H(e^{j[\omega+2k\pi]})$). In fact, we may view the previous expression as the exponential Fourier series expansion for $H(e^{j\omega})$, with $h(k)$ as the Fourier series coefficients. Consequently, the unit impulse response $h(k)$ is related to $H(e^{j\omega})$ through the integral expression

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}\,d\omega$$

1.3.2. Comments on Symmetry Properties

For LTI systems with real-valued impulse response, the magnitude response, phase responses, the real component of and the imaginary component of $H(e^{j\omega})$ possess these symmetry properties:

The real component of $H(e^{j\omega})$: $\mathrm{Re}\left[H(e^{-j\omega})\right] = \mathrm{Re}\left[H(e^{j\omega})\right]$ (even function of ω periodic with period 2π)

The imaginary component of $H(e^{j\omega})$: $\mathrm{Im}\left[H(e^{-j\omega})\right] = -\mathrm{Im}\left[H(e^{j\omega})\right]$ (odd function of ω periodic with period 2π

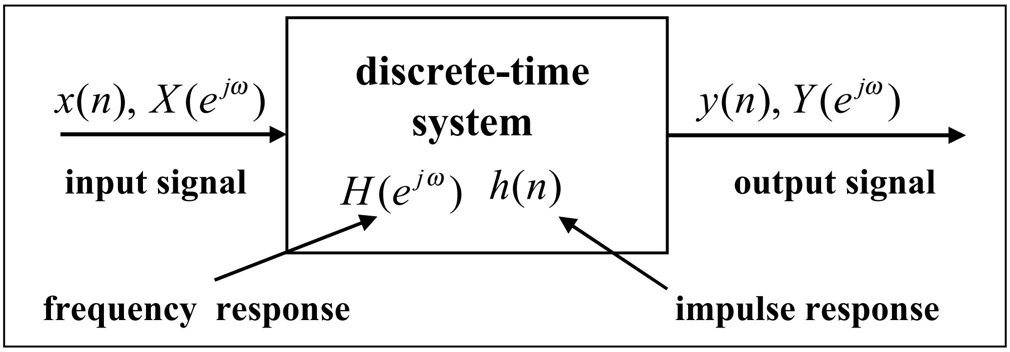
The magnitude response of $H(e^{j\omega})$: $\left|H(e^{-j\omega})\right| = \left|H(e^{j\omega})\right|$ (even function of ω periodic with period 2π)

The phase response of $H(e^{j\omega})$: $\arg\left[H(e^{-j\omega})\right] = -\arg\left[H(e^{-j\omega})\right]$ (odd function of ω periodic with period 2π)

Consequence:

If we known $\left|H(e^{j\omega})\right|$ and $\phi(\omega)$ for $0 \le \omega \le \pi$, we can describe these functions (i.e. also $H(e^{j\omega})$) for all values of ω.

1.3.3. Comments on Fourier Transform of Discrete Signals and Frequency-Domain Description of LTI Systems



The input signal $x(n)$: $X(e^{j\omega}) = \sum_{k=-\infty}^{\infty} x(k)e^{-j\omega k}$, $x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n}\, d\omega$

The output signal $y(n)$: $Y(e^{j\omega}) = \sum_{k=-\infty}^{\infty} y(k)e^{-j\omega k}$, $y(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y(e^{j\omega})e^{j\omega n}\, d\omega$

The impulse response $h(n)$: $H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}$, $h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n}\, d\omega$

Frequency-Domain Description of LTI System: $Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega})$

1.3.4. Comments on Normalized Frequency

It is often desirable to express the frequency response of a sequence $h(n) = h(nT)$ in terms of units of frequency that involve sampling interval T. In this case, the expression

$$H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k) e^{-j\omega k} \ , \ h(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega}) e^{j\omega n} \, d\omega$$

are modified to the form:

$$H(e^{j\omega T}) = \sum_{k=-\infty}^{\infty} h(kT) e^{-j\omega k T} \ , \ h(nT) = \frac{T}{2\pi} \int_{-\pi/T}^{\pi/T} H(e^{j\omega T}) e^{j\omega n T} \, d\omega$$

$H(e^{j\omega T})$ is periodic with period $2\pi / T = 2\pi F$, where F is sampling frequency.

Solution: normalized frequency approach: $F/2 \to \pi$.

Example:

$F = 100\,kHz$, $F/2 = 50\,kHz$, $50\,kHz \to \pi$

$f_1 = 20\,kHz$, $\omega_1 = \dfrac{20\pi}{50} = \dfrac{2\pi}{5} = 0.4\pi$

$f_2 = 25\,kHz$, $\omega_2 = \dfrac{25\pi}{50} = \dfrac{\pi}{2} = 0.5\pi$

Example: