4.2.3. Uniform Frequency-Sampling Method 2. Nonrecursive FIR Filter Design by DFT Applications

The frequency sampling method for FIR filter design is basically the technique described in the previous sections. Recall that we specify the desired frequency response at a set of equally spaced frequencies (see e.g. Table 4.2.3.1.) and solved for the unit sample response *h(n)* of the FIR filter from these equally spaced specification. We also recall that it is desirable to optimize the frequency specification in the transition band of the filter. This optimization can be accomplished numerically on a digital computer by means of linear programming techniques. In his section we consider the frequency sampling method in greater depth and generality.

Let us begin with the frequency response of the FIR filter

$$y(n) = \sum_{k=0}^{M-1} h(k)x(n-k)$$

$$H(e^{j\omega}) = \sum_{k=0}^{M-1} h(k)e^{-j\omega k}$$

Suppose that we specify the frequency response of the filter at the frequencies

$$\omega_n = \frac{2\pi n}{M}$$

Then, we obtain

$$H(e^{j\omega_n}) = H(\omega_n) = H\left(\frac{2\pi n}{M}\right) = H(n)$$

and

$$H(n) = \sum_{k=0}^{M-1} h(k)e^{-j\omega_n} = \sum_{k=0}^{M-1} h(k)e^{-j\frac{2\pi nk}{M}} \quad \text{for} \quad n = 0,1,2,\ldots,M-1$$

It is simple matter to identify the previous expression as the Discrete Fourier Transform (DFT) of *h(n),* then

$$H(n) = DFT\left[h(n)\right]$$

Thus, we obtain

$$h(n) = DFT^{-1}\left[H(n)\right]$$

or

$$h(k) = \frac{1}{M}\sum_{n=0}^{M-1} H(n)e^{j\frac{2\pi nk}{M}} \quad \text{for} \quad k = 0,1,2,\ldots,M-1$$

This expression allows us to compute the values of the unit sample response *h(n)* from the specification of the frequency samples *H(k)*. In order to do it, it is necessary to specify *H(k)* correctly. In the next, we will present how to define *H(k)* in order to apply for computation *h(n)*.

Since *h(n)* is real it is easily established that

$$\overline{H(M-n)} = \overline{\sum_{k=0}^{M-1} h(k) e^{-j\frac{2\pi k(M-n)}{M}}} = \sum_{k=0}^{M-1} h(k) e^{j\frac{2\pi k(M-n)}{M}} = \sum_{k=0}^{M-1} h(k) e^{j\frac{2\pi kM}{M}} e^{j\frac{2\pi k(-n)}{M}} =$$

$$= \sum_{k=0}^{M-1} h(k) e^{j2\pi k} e^{-j\frac{2\pi kn}{M}} = \sum_{k=0}^{M-1} h(k) e^{-j\frac{2\pi kn}{M}} = H(n)$$

i.e.

$$H(n) = \overline{H(M-n)} \text{ for } n = 1, 2, \ldots, M-1$$

or

$$H(M-n) = \overline{H(n)} \text{ for } n = 1, 2, \ldots, M-1$$

Recall that we have shown the validity of the following expressions for the linear phase FIR digital filters:

$$H(e^{j\omega_n}) = H_r(\omega_n) e^{-j\omega_n \frac{M-1}{2}}$$

Then, we can express $H(e^{j\omega_n})$ as

$$H(e^{j\omega_n}) = H(n) = (-1)^n H_r\left(\frac{2\pi n}{M}\right) e^{j\frac{\pi n}{M}}$$

Proof:

$$H(e^{j\omega_n}) = H_r(\omega_n) e^{-j\omega_n \frac{M-1}{2}} = H_r\left(\frac{2\pi n}{M}\right) e^{-j\frac{2\pi n}{M}\frac{M-1}{2}} = H_r\left(\frac{2\pi n}{M}\right) e^{-j\pi n \frac{M-1}{M}} =$$

$$= H_r\left(\frac{2\pi n}{M}\right) e^{-j\frac{\pi n M}{M}} e^{j\frac{\pi n}{M}} = H_r\left(\frac{2\pi n}{M}\right) e^{-j\pi n} e^{j\frac{\pi n}{M}} = (-1)^n H_r\left(\frac{2\pi n}{M}\right) e^{j\frac{\pi n}{M}}$$

With regards to the previous expressions we can design the linear phase FIR filter by using the following steps (*Uniform Frequency-Sampling Method 2.*):

1. Specification of $H(n)$ by:

$$H(n) = (-1)^n H_r\left(\frac{2\pi n}{M}\right) e^{j\frac{\pi n}{M}} \text{ and taking into account } H(M-n) = \overline{H(n)} \text{ for } n = 1, 2, \ldots, M-1$$

2. Computation of *h(n)* by:

$$h(k) = DFT^{-1}\left[H(n)\right] = \frac{1}{M} \sum_{n=0}^{M-1} H(n) e^{j\frac{2\pi nk}{M}} \text{ for } k = 0, 1, 2, \ldots, M-1$$

Example:

Determine the coefficients *h(n)* of a linear phase FIR filter of length *M=15* frequency response of which satisfies the condition:

$$H_r\left(\frac{2\pi k}{15}\right)=\begin{cases}1 & k=0,1,2,3\\ 0 & k=4,5,6,7\end{cases}$$

Solution:

$$H(n)=(-1)^n H_r\left(\frac{2\pi n}{M}\right)e^{j\frac{\pi n}{M}}=(-1)^n H_r\left(\frac{2\pi n}{15}\right)e^{j\frac{\pi n}{15}}$$ and taking into account $H(M-n)=\overline{H(n)}$ for $n=1,2,\ldots,M-1$

$n=0\quad H(0)=1$

$n=1\quad H(1)=(-1)^1 H_r\left(\frac{1.2\pi}{15}\right)e^{j\frac{1\pi}{15}}=-e^{j\frac{\pi}{15}}\qquad H(14)=H(15-1)=\overline{H(1)}=-e^{-j\frac{\pi}{15}}$

$n=2\quad H(2)=(-1)^2 H_r\left(\frac{2.2\pi}{15}\right)e^{j\frac{2\pi}{15}}=e^{j\frac{2\pi}{15}}\qquad H(13)=H(15-2)=\overline{H(2)}=e^{-j\frac{2\pi}{15}}$

$n=3\quad H(3)=(-1)^3 H_r\left(\frac{3.2\pi}{15}\right)e^{j\frac{3\pi}{15}}=-e^{j\frac{3\pi}{15}}\qquad H(12)=H(15-3)=\overline{H(3)}=-e^{-j\frac{3\pi}{15}}$

$n=4\quad H(4)=0\qquad H(11)=H(15-4)=\overline{H(4)}=0$

$n=5\quad H(5)=0\qquad H(10)=H(15-5)=\overline{H(5)}=0$

$n=6\quad H(6)=0\qquad H(9)=H(15-6)=\overline{H(6)}=0$

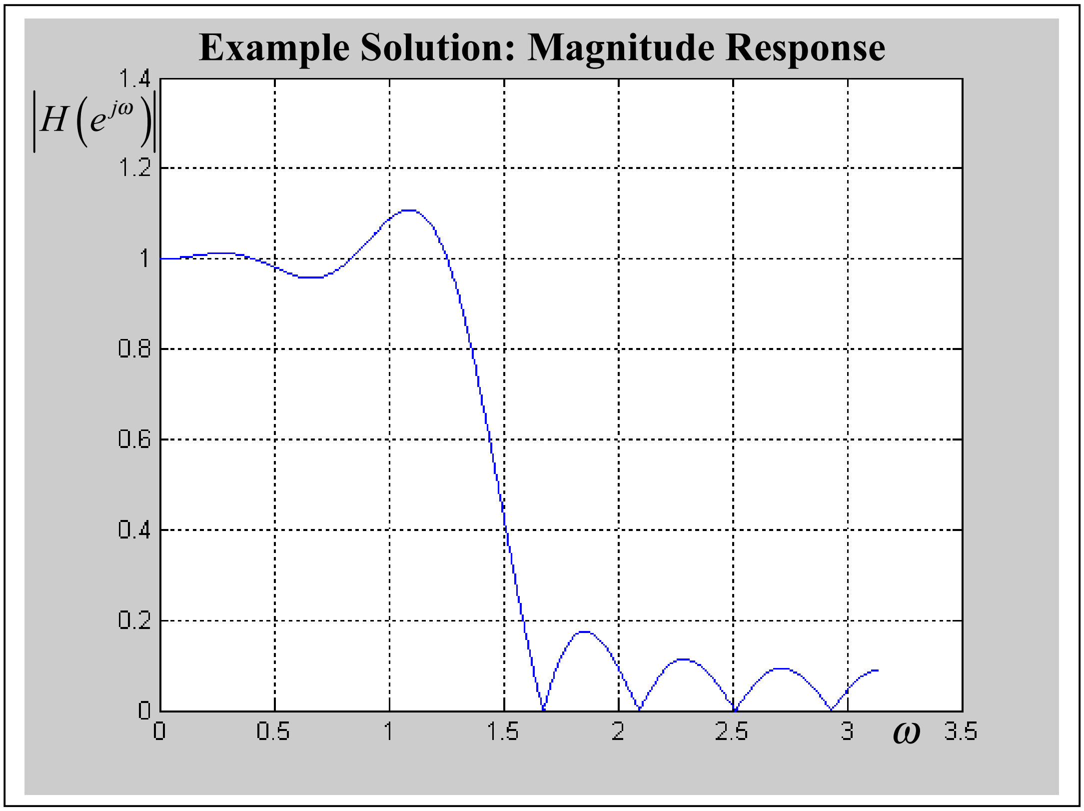
$n=7\quad H(7)=0\qquad H(8)=H(15-7)=\overline{H(7)}=0$

$$h(k)=DFT^{-1}\left[H(n)\right]=\frac{1}{15}\sum_{n=0}^{15-1}H(n)e^{j\frac{2\pi nk}{15}}=\frac{1}{15}\sum_{n=0}^{14}H(n)e^{j\frac{2\pi nk}{15}}\quad \text{for } k=0,1,2,\ldots,14$$

Then, by using MATLAB (function *ifft.m*) the following filter coefficients can be obtained:

h(0)=h(14)= -0.0498
h(1)=h(13)= 0.0412
h(2)=h(12)= 0.0667
h(3)=h(11)= -0.0365
h(4)=h(10)= -0.1079
h(5)=h(9)= 0.0341
h(6)=h(8)= 0.3189
h(7)= 0.4667

End.



4.2.4. Uniform Frequency-Sampling Method 3. Nonrecursive FIR Filter Design by Direct Computation of Unit Sample Response

In the previous chapter we have developed the method for linear phase FIR filter design by the direct application of DFT. In this chapter, we will derive a number of expression sets for the direct design of linear phase FIR filter.

We have shown that

$$H(n) = (-1)^n H_r\left(\frac{2\pi n}{M}\right) e^{j\frac{\pi n}{M}}$$

It is more convenient, however, to define the samples *H(k)* of the frequency response for the FIR filter as

$$H(n) = G(n) e^{j\frac{\pi n}{M}}$$

where it is easily verified that

$$G(n) = (-1)^n H_r\left(\frac{2\pi n}{M}\right)$$

Since $H_r\left(\frac{2\pi n}{M}\right)$ is purely real, so is the sequence of frequency samples *G(k)*. Moreover, the condition

$H(M-n)=\overline{H(n)}$ leads to the result that

$$G(n)=-G(M-n) \text{ for } n=1,2,\ldots,M-1$$

When *M* is even, this condition requires that

$$G\left(\frac{M}{2}\right)=0$$

In other words, the frequency sample at $\omega=\pi$ must be zero when *M* is even.

Proof:

$H(n)=\overline{H(M-n)}$, $H(n)=G(n)e^{j\frac{\pi n}{M}}$

Then,

$$H(n)=\overline{G(M-n)e^{j\frac{\pi(M-n)}{M}}}=G(M-n)e^{-j\frac{\pi(M-n)}{M}}=G(M-n)e^{-j\frac{\pi M}{M}}e^{j\frac{\pi n}{M}}=-G(M-n)e^{j\frac{\pi n}{M}}$$

$$H(n)=G(n)e^{j\frac{\pi n}{M}}$$

Then,

$$-G(M-n)e^{j\frac{\pi n}{M}}=G(n)e^{j\frac{\pi n}{M}}$$

It follows from the previous expression that

$$G(n)=-G(M-n) \text{ for } n=1,2,\ldots,M-1$$

In the view of the symmetry for the real-valued frequency samples *G(n),* it is a simple matter to determine an expression for the unit sample response *h(n)* of the FIR filter in terms of *G(n)*. If *M* is even, we have

$$h(n)=\frac{1}{M}\sum_{k=0}^{M-1}H(k)e^{j\frac{2\pi kn}{M}}=\frac{1}{M}\sum_{k=0}^{M-1}G(k)e^{j\frac{\pi k}{M}}e^{j\frac{2\pi kn}{M}}=\frac{1}{M}\sum_{k=0}^{M-1}G(k)e^{j\frac{\pi k(2n+1)}{M}}=$$

$$=\frac{1}{M}\left\{G(0)+\sum_{k=1}^{\frac{M}{2}-1}G(k)\left[e^{j\frac{\pi k(2n+1)}{M}}-e^{j\frac{\pi(M-k)(2n+1)}{M}}\right]\right\}$$

Because,

$$e^{j\frac{\pi(M-k)(2n+1)}{M}}=e^{j\frac{\pi}{M}[(M-k)2n+(M-k)]}=e^{j\frac{\pi}{M}(M-k)2n}e^{j\frac{\pi}{M}(M-k)}=$$

$$=e^{j\frac{2\pi nM}{M}}e^{-j\frac{2\pi kn}{M}}e^{j\frac{\pi M}{M}}e^{-j\frac{\pi k}{M}}=-e^{-j\frac{\pi k}{M}(2n+1)}$$

we can get

$$h(n)=\frac{1}{M}\left\{G(0)+2\sum_{k=1}^{\frac{M}{2}-1}G(k)\left[\frac{e^{j\frac{\pi k(2n+1)}{M}}+e^{-j\frac{\pi k(2n+1)}{M}}}{2}\right]\right\}$$

and

$$h(n)=\frac{1}{M}\left\{G(0)+2\sum_{k=1}^{\frac{M}{2}-1}G(k)\cos\frac{\pi k}{M}(2n+1)\right\}\text{ for } n=0,1,2,\ldots,M-1$$

Thus we can compute *h(n)* directly from the specified frequency samples *G(k)* or equivalently, *H(k)* (*method 3*). This approach eliminates the need for a matrix inversion (*method 1*) or for direct DFT computation (*method 2*) for determining the unit sample response of the linear phase FIR filter from its frequency-domain specification.

The next Table summarizes all important relationships between *h(n)* and the frequency response samples that can be derived for all of the four types of linear phase FIR filters. The details concerning the derivation of these expressions can be found in the references.

Table 4.2.4.1. Summary on the Uniform Frequency-Sampling Method 3.

Unit Sample Response: Symmetric $\alpha=0$

$$H(k)=G(k)e^{j\pi k/M},\ k=0,1,\ldots,M-1$$

$$G(k)=(-1)^k H_r\left(\frac{2\pi k}{M}\right)$$

$$G(k)=-G(M-k)$$

$$h(n)=\frac{1}{M}\left\{G(0)+2\sum_{k=1}^{U}G(k)\cos\frac{\pi k}{M}(2n+1)\right\}$$

$$U=\frac{M-1}{2}\ \textit{for } M \textit{ odd} \qquad U=\frac{M}{2}-1\ \textit{for } M \textit{ even}$$

Unit Sample Response: Symmetric $\alpha = \frac{1}{2}$

$$H\left(k+\frac{1}{2}\right) = G\left(k+\frac{1}{2}\right)e^{-j\pi/2}e^{j\pi(2k+1)/2M}$$

$$G\left(k+\frac{1}{2}\right) = (-1)^k H_r\left[\frac{2\pi}{M}\left(k+\frac{1}{2}\right)\right]$$

$$G\left(k+\frac{1}{2}\right) = G\left(M-k-\frac{1}{2}\right)$$

$$h(n) = \frac{2}{M}\sum_{k=0}^{U} G\left(k+\frac{1}{2}\right)\sin\frac{2\pi}{M}\left(k+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)$$

Unit Sample Response: Antisymmetric $\alpha = 0$

$$H(k) = G(k)e^{j\pi/2}e^{j\pi k/M},\ \ k = 0,1,\ldots,M-1$$

$$G(k) = (-1)^k H_r\left(\frac{2\pi k}{M}\right)$$

$$G(k) = G(M-k)$$

$$h(n) = -\frac{2}{M}\sum_{k=1}^{(M-1)/2} G(k)\sin\frac{\pi k}{M}(2n+1)\ \ \textit{for}\ \ M\ \textit{odd}$$

$$h(n) = \frac{1}{M}\left\{(-1)^{n+1}G(M/2) - 2\sum_{k=1}^{(M/2)-1} G(k)\sin\frac{\pi k}{M}(2n+1)\right\}\ \ \textit{for}\ \ M\ \textit{even}$$

Unit Sample Response: Antisymmetric $\alpha = \frac{1}{2}$

$$H\left(k+\frac{1}{2}\right) = G\left(k+\frac{1}{2}\right)e^{j\pi(2k+1)/2M}$$

$$G\left(k+\frac{1}{2}\right) = (-1)^k H_r\left[\frac{2\pi}{M}\left(k+\frac{1}{2}\right)\right]$$

$$G\left(k+\frac{1}{2}\right) = -G\left(M-k-\frac{1}{2}\right);\ \ G\left(\frac{M}{2}\right) = 0\ \textit{for odd}$$

$$h(n)=\frac{2}{M}\sum_{k=0}^{V}G\left(k+\frac{1}{2}\right)\cos\frac{2\pi}{M}\left(k+\frac{1}{2}\right)\left(n+\frac{1}{2}\right)$$

$$V=\frac{M-1}{2}\ \ for\ M\ odd \qquad V=\frac{M}{2}-1\ \ for\ M\ even$$

Example:

Determine the coefficients *h(n)* of a linear phase FIR filter of length *M=15* symmetric frequency response of which satisfies the condition:

$$H_r\left(\frac{2\pi k}{15}\right)=\begin{cases}1 & k=0,1,2,3\\ 0 & k=4,5,6,7\end{cases}$$

Solution:

Unit Sample Response: Symmetric $\alpha=0$, *M* odd

$$G(k)=(-1)^k\,H_r\left(\frac{2\pi k}{M}\right)=(-1)^k\,H_r\left(\frac{2\pi k}{15}\right)$$

$$G(0)=1$$

$$G(1)=(-1)^1\,H_r\left(\frac{1.2\pi}{15}\right)=-1$$

$$G(2)=(-1)^2\,H_r\left(\frac{2.2\pi}{15}\right)=1$$

$$G(3)=(-1)^3\,H_r\left(\frac{3.2\pi}{15}\right)=-1$$

$$G(4)=0 \qquad G(5)=0 \qquad G(6)=0 \qquad G(7)=0$$

$$U=\frac{M-1}{2}=\frac{15-1}{2}=\frac{14}{2}=7$$

$$h(n)=\frac{1}{15}\left\{G(0)+2\sum_{k=1}^{7}G(k)\cos\frac{\pi k}{15}(2n+1)\right\}$$

$$h(n)=\frac{1}{15}\left\{G(0)+2\left[G(1)\cos\frac{\pi}{15}(2n+1)+G(2)\cos\frac{2\pi}{15}(2n+1)+G(3)\cos\frac{3\pi}{15}(2n+1)\right]\right\}$$

$$h(n)=\frac{1}{15}\left\{1+2\left[-\cos\frac{\pi}{15}(2n+1)+\cos\frac{2\pi}{15}(2n+1)-\cos\frac{3\pi}{15}(2n+1)\right]\right\}$$

$n = 0$

$$h(0) = \frac{1}{15}\left\{1 + 2\left[-\cos\frac{\pi}{15} + \cos\frac{2\pi}{15} - \cos\frac{3\pi}{15}\right]\right\} =$$

$$= \frac{1}{15}\{1 + 2[-0.9781 + 0.9135 - 0.8090]\} =$$

$$= \frac{1}{15}[1 + 2(-0.8736)] = -0.0498$$

Etc.

h(0)=h(14)= -0.0498
h(1)=h(13)= 0.0412
h(2)=h(12)= 0.0667
h(3)=h(11)= -0.0365
h(4)=h(10)= -0.1079
h(5)=h(9)= 0.0341
h(6)=h(8)= 0.3189
h(7)= 0.4667

End.

4.2.5. Uniform Frequency-Sampling Method 4. Recursive FIR Filter Design

An FIR filter can be uniquely specified by giving either the impulse response coefficients *h(n)* or, equivalently, the DFT coefficients *H(k)*. The reader will recall that these sequences are related by the following DFT relations

$$H(n) = \sum_{k=0}^{M-1} h(k)e^{-j\frac{2\pi nk}{M}} = DFT[h(k)] \text{ for } n = 0,1,2,\ldots,M-1$$

$$h(k) = \frac{1}{M}\sum_{n=0}^{M-1} H(n)e^{j\frac{2\pi nk}{M}} = DFT^{-1}[H(n)] \text{ for } k = 0,1,2,\ldots,M-1$$

It has also been shown that the DFT samples *H(n)* for an FIR sequence can be regarded as samples of the filter's z-transform, evaluated at *N* points equally spaced around the unit circle; i.e.

$$H(k) = H(z)\ for\ z = e^{j(2\pi/N)k}$$

Thus the transfer function of an FIR filter can easily be expressed in terms of DFT coefficients of *h(n)*:

$$H(z) = \sum_{k=0}^{M-1} h(k)z^{-k} = \sum_{k=0}^{M-1} z^{-k}\left[\frac{1}{M}\sum_{n=0}^{M-1} H(n)e^{j\frac{2\pi kn}{M}}\right] =$$

$$= \frac{1}{M}\sum_{n=0}^{M-1} H(n)\sum_{k=0}^{M-1}\left[e^{j2\pi n/M}z^{-1}\right]^k = \frac{1}{M}\sum_{n=0}^{M-1} H(n)\frac{1-\left(e^{j2\pi n/M}z^{-1}\right)^M}{1-e^{j2\pi n/M}z^{-1}} =$$

$$= \frac{1}{M}\sum_{n=0}^{M-1} H(n)\frac{1-e^{j2\pi n}z^{-M}}{1-e^{j2\pi n/M}z^{-1}} = \frac{1}{M}\sum_{n=0}^{M-1} H(n)\frac{1-z^{-M}}{1-e^{j2\pi n/M}z^{-1}} =$$

$$= \frac{1-z^{-M}}{M}\sum_{n=0}^{M-1}\frac{H(n)}{1-e^{j2\pi n/M}z^{-1}}$$

i.e.

$$H(z) = \frac{1-z^{-M}}{M}\sum_{n=0}^{M-1}\frac{H(n)}{1-e^{j2\pi n/M}z^{-1}}$$

The transfer (system) function *H(z)* can be expressed also in the form

$$H(z) = H_1(z)H_2(z)$$

where

$$H_1(z) = \frac{1-z^{-M}}{M} \qquad H_2(z) = \sum_{n=0}^{M-1}\frac{H(n)}{1-e^{j2\pi n/M}z^{-1}}$$

With regard to these expressions, the filter with transfer (system) function *H(z)* can be interpreted as a cascade of a comb filter having the transfer (system) function $H_1(z)$ and a parallel bank of single-pole filters having the transfer (system) function c.

The zeros of the comb filter $H_1(z)$ coincide and thus cancel the poles of the parallel bank of filters $H_2(z)$, so that in effect the transfer (system) function of the cascade connection $H_1(z)H_2(z)$ contains no poles, but it is in fact FIR filter.

Alternative form to

$$H(z) = \frac{1-z^{-M}}{M}\sum_{n=0}^{M-1}\frac{H(n)}{1-e^{j2\pi n/M}z^{-1}}$$

can be obtained by combining a pair of complex-conjugate poles of *H(z)* or two real-valued poles to for two-pole filters with real valued coefficients and taking into account $H(n) = \overline{H(M-n)}$. Then, we can obtain the transfer (system) function for $H_2(z)$ in the forms

$$H_2(z) = \frac{H(0)}{1-z^{-1}} + \sum_{k=1}^{\frac{M-1}{2}}\frac{A(k)-B(k)z^{-1}}{1-2\cos(2\pi k/M)z^{-1}+z^{-2}} \quad \textit{for M odd}$$

$$H_2(z) = \frac{H(0)}{1-z^{-1}} + \frac{H(M/2)}{1+z^{-1}} + \sum_{k=1}^{\frac{M}{2}-1}\frac{A(k)-B(k)z^{-1}}{1-2\cos(2\pi k/M)z^{-1}+z^{-2}} \quad \textit{for M even}$$

where

$$A(k) = H(k) + H(M-k)$$

$$B(k) = H(k)e^{-j2\pi k/M} + H(M-k)e^{j2\pi k/M}$$

are the real-valued coefficients.

Proof:

$$\frac{H(k)}{1-e^{j2\pi k/M}z^{-1}} + \frac{H(M-k)}{1-e^{j2\pi(M-k)/M}z^{-1}} = \frac{H(k)}{1-e^{j2\pi k/M}z^{-1}} + \frac{H(M-k)}{1-e^{j2\pi M/M}e^{-j2\pi k/M}z^{-1}} =$$

$$= \frac{H(k)}{1-e^{j2\pi k/M}z^{-1}} + \frac{H(M-k)}{1-e^{-j2\pi k/M}z^{-1}} = \frac{H(k)\left(1-e^{-j2\pi k/M}z^{-1}\right) + H(M-k)\left(1-e^{j2\pi k/M}z^{-1}\right)}{\left(1-e^{j2\pi k/M}z^{-1}\right)\left(1-e^{-j2\pi k/M}z^{-1}\right)} =$$

$$= \frac{\left[H(k)+H(M-k)\right] - \left[H(k)e^{-j2\pi k/M} + H(M-k)e^{j2\pi k/M}\right]z^{-1}}{1-2\dfrac{e^{j2\pi k/M}+e^{-j2\pi k/M}}{2}z^{-1}+z^{-2}} =$$

$$= \frac{\left[H(k)+H(M-k)\right] - \left[H(k)e^{-j2\pi k/M} + H(M-k)e^{j2\pi k/M}\right]z^{-1}}{1-2\cos\dfrac{2\pi k}{M}z^{-1}+z^{-2}} =$$

$$= \frac{A(k)-B(k)z^{-1}}{1-2\cos\dfrac{2\pi k}{M}z^{-1}+z^{-2}}$$

Comments

Because, $H(n) = \overline{H(M-n)}$

$$A(k) = H(k) + H(M-k) = H(k) + \overline{H(k)} = 2\,\mathrm{Re}\left[H(h)\right] \in R$$

Let us assume that

$$\phi(\omega) = \arg\left[H(k)\right]$$

Then

$$B(k) = H(k)e^{-j2\pi k/M} + H(M-k)e^{j2\pi k/M} =$$

$$= \left|H(k)\right| e^{j\phi(\omega)} e^{-j2\pi k/M} + \overline{H(k)} e^{j2\pi k/M} =$$

$$= \left|H(k)\right| e^{j\phi(\omega)} e^{-j2\pi k/M} + \left|H(k)\right| e^{-j\phi(\omega)} e^{j2\pi k/M} =$$

$$= 2\left|H(k)\right| \frac{e^{j[\phi(\omega)-2\pi k/M]} + e^{-j[\phi(\omega)-2\pi k/M]}}{2} =$$

$$= 2\left|H(k)\right| \cos\left(\phi(\omega) - 2\pi k/M\right) \in R$$

There is a potential problem in the frequency sampling realization of the recursive FIR linear phase filter. As indicated previous expressions, these realizations introduce poles and zeros at equally spaced points on the unit circle. In the ideal situation, the zeros cancel the poles and, consequently, the actual zeros of *H(z)* are determined by the selection of the frequency samples *H(k)*. In a practical implementation of the frequency-sampling realization, however, quantization effects preclude a perfect cancellation of the poles and zeros. In fact, the locations of poles on the unit circle provide no damping of the round-off noise that introduced in the As result, such noise tends to increase with time and, ultimately, may destroy computations the normal operation of the filter.

To mitigate this problem, we can move both the poles and zeros from the unit circle to a circle just inside the unit circle, say at radius $r=1-x$, where x is a very small number. Thus the system function of the linear phase FIR filter becomes

$$H(z) = \frac{1 - r^M z^{-M}}{M} \sum_{n=0}^{M-1} \frac{H(n)}{1 - re^{j2\pi n/M} z^{-1}}$$

The corresponding two-pole filter realization can be modeled accordingly. The damping provided by selecting $r<1$ ensures that round-off noise will be bounded and thus instability is avoided.